Linear programming

Linear programming problem is the branch of applied mathematics that deals with solving optimization problem of a particular function form.

Linear programming (**LP**), also called **linear optimization**, is a method to achieve the best outcome (such as maximum profit or lowest cost) in a mathematical model whose requirements are represented by linear relationships. Linear programming is a special case of mathematical programming (also known as mathematical optimization).

Linear Programming deals with the problem of optimizing a linear objective function subject to linear equality and inequality constraints on the decision variables. Linear programming has many practical applications (in transportation, production planning, ...). It is also the building block for combinatorial optimization.

Graphical Solution of Linear Programming Problems

The graphical method is one of the easiest and most important methods for solving Linear Programming Problems. In Graphical Solution of Linear Programming, we use graphs to solve LPP. We can solve a wide variety of problems using Linear programming in different sectors, but it is generally used for problems in which we have to maximize profit, minimize cost, or minimize the use of resources [1]. Here are the steps involved in solving a linear programming problem using the graphical method:

1. Formulate the mathematical model by decoding the given situations or constraints into equations or inequalities.
2. Graph the system of constraints. This will give the feasible set.
3. Find each vertex (corner point) of the feasible set.
4. Substitute each vertex into the objective function to determine which vertex optimizes the objective function.
5. State the solution to the problem [2].

We can solve linear programming problems using two different methods are,

1. Corner Point Methods
2. Iso-Cost Methods

Corner Point Methods

To solve the problem using the corner point method you need to follow the following steps:

Step 1: Create mathematical formulation from the given problem. If not given.
Step 2: Now plot the graph using the given constraints and find the feasible region.
Step 3: Find the coordinates of the feasible region(vertices) that we get from step 2.
Step 4: Now evaluate the objective function at each corner point of the feasible region. Assume N and n denotes the largest and smallest values of these points.

Step 5: If the feasible region is bounded then N and n are the maximum and minimum value of the objective function. Or if the feasible region is unbounded then:

- N is the maximum value of the objective function if the open half plan is got by the ax + by > N has no common point to the feasible region. Otherwise, the objective function has no solution.
- n is the minimum value of the objective function if the open half plan is got by the ax + by < n has no common point to the feasible region. Otherwise, the objective function has no solution.

Examples on LPP using Corner Point Methods

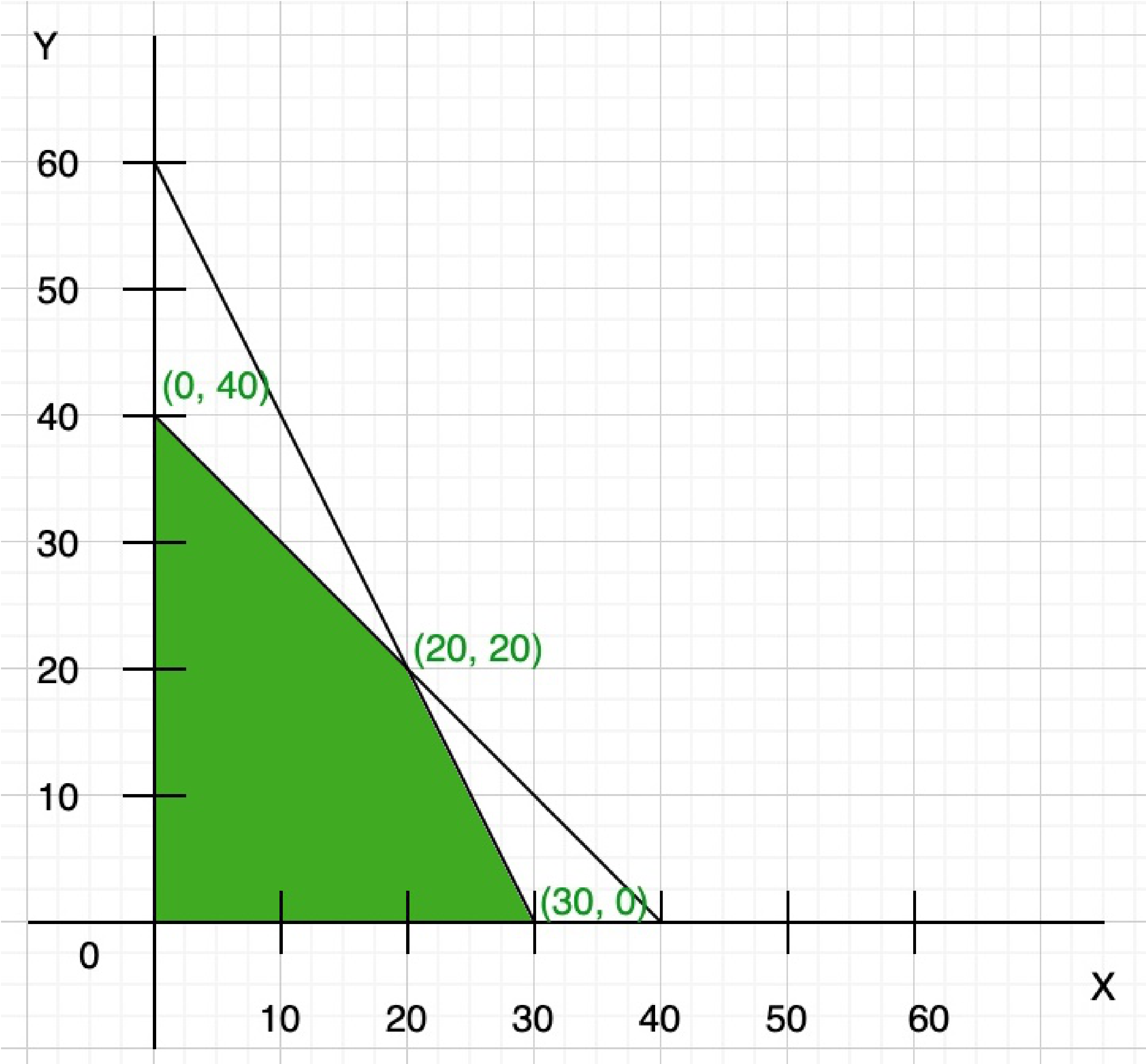
Example 1: Solve the given linear programming problems graphically:
Maximize: Z = 8x + y
Constraints are,

- **x + y ≤ 40**
- **2x + y ≤ 60**
- **x ≥ 0, y ≥ 0**

Solution:
Step 1: Constraints are,

- x + y ≤ 40
- 2x + y ≤ 60
- x ≥ 0, y ≥ 0

Step 2: Draw the graph using these constraints.



Here both the constraints are less than or equal to, so they satisfy the below region (towards origin). You can find the vertex of feasible region by graph, or you can calculate using the given constraints:

x + y = 40 …(i)

2x + y = 60 …(ii)

Now multiply eq(i) by 2 and then subtract both eq(i) and (ii), we get

y = 20

Now put the value of y in any of the equations, we get

x = 20

So the third point of the feasible region is (20, 20)

Iso-Cost Methods

The term iso-cost or iso-profit method provides the combination of points that produces the same cost/profit as any other combination on the same line. This is done by plotting lines parallel to the slope of the equation.

To solve the problem using Iso-cost method you need to follow the following steps:

Step 1: Create mathematical formulation from the given problem. If not given.

Step 2: Now plot the graph using the given constraints and find the feasible region.

Step 3: Now find the coordinates of the feasible region that we get from step 2.

Step 4: Find the convenient value of Z(objective function) and draw the line of this objective function.

Step 5: If the objective function is maximum type then draw a line which is parallel to the objective function line and this line is farthest from the origin and only has one common point to the feasible region. Or if the objective function is minimum type then draw a line which is parallel to the objective function line and this line is nearest from the origin and has at least one common point to the feasible region.

Step 6: Now get the coordinates of the common point that we find in step 5. Now, this point is used to find the optimal solution and the value of the objective function.

Solved Examples of Graphical Solution of LPP

Example 1: Solve the given linear programming problems graphically:

Maximize: Z = 50x + 15y

Constraints are,

- **$5x + y \leq 100$**
- **$x + y \leq 50$**
- **$x \geq 0, y \geq 0$**

Solution:

Given,

- $5x + y \leq 100$
- $x + y \leq 50$
- $x \geq 0, y \geq 0$

Step 1: Finding points

We can also write as

5x + y = 100….(i)

x + y = 50….(ii)

Now we find the points

so we take eq(i), now in this equation

When x = 0, y = 100

When y = 0, x = 20

So, points are (0, 100) and (20, 0)

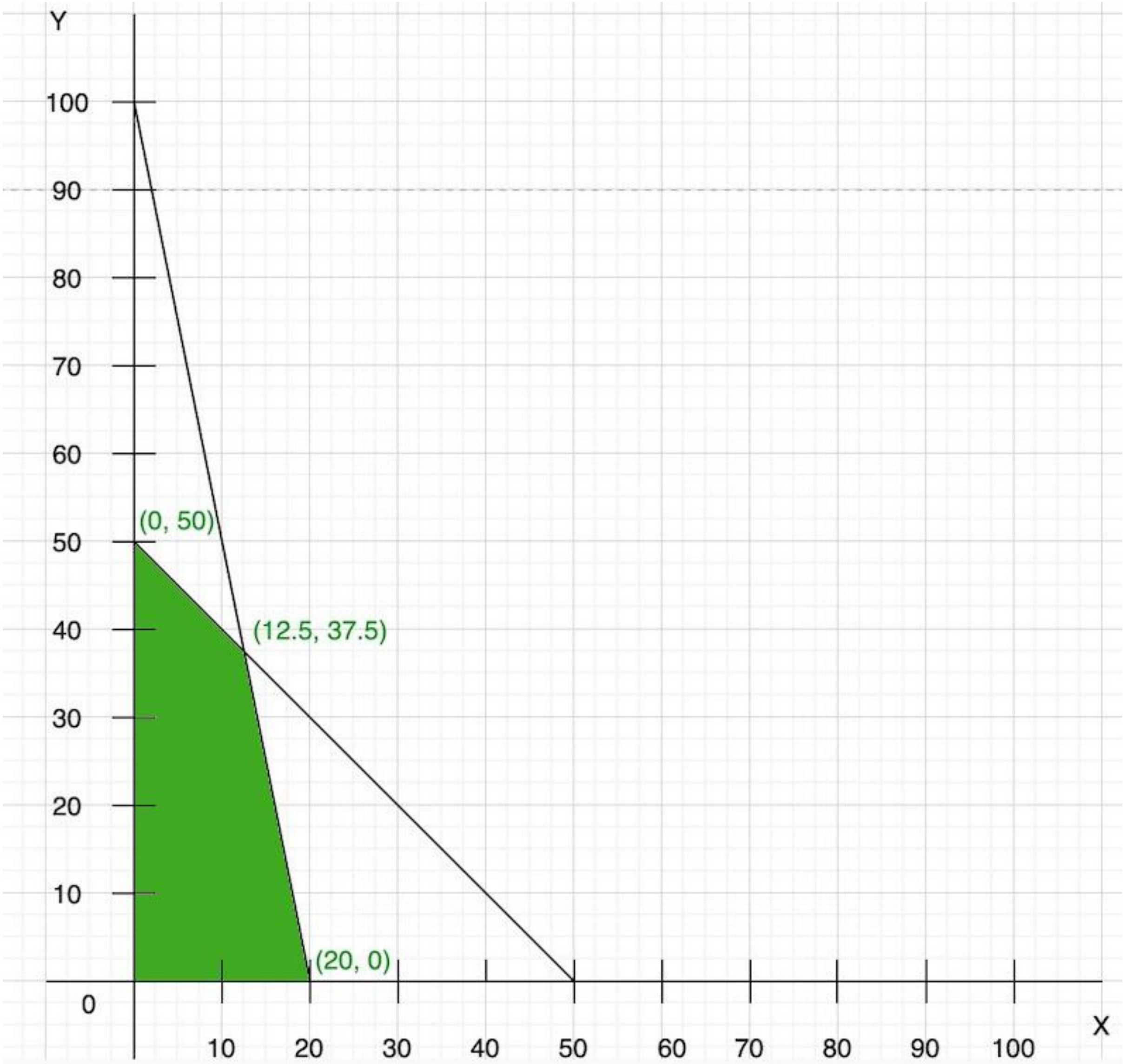
Similarly, in eq(ii)

When x = 0, y = 50

When y = 0, x = 50

So, points are (0, 50) and (50, 0)

Step 2: Now plot these points in the graph and find the feasible region.



Step 3: Now we find the convenient value of Z(objective function)

So, to find the convenient value of Z, we have to take the lcm of coefficient of 50x + 15y, i.e., 150. So, the value of Z is the multiple of 150, i.e., 300. Hence,

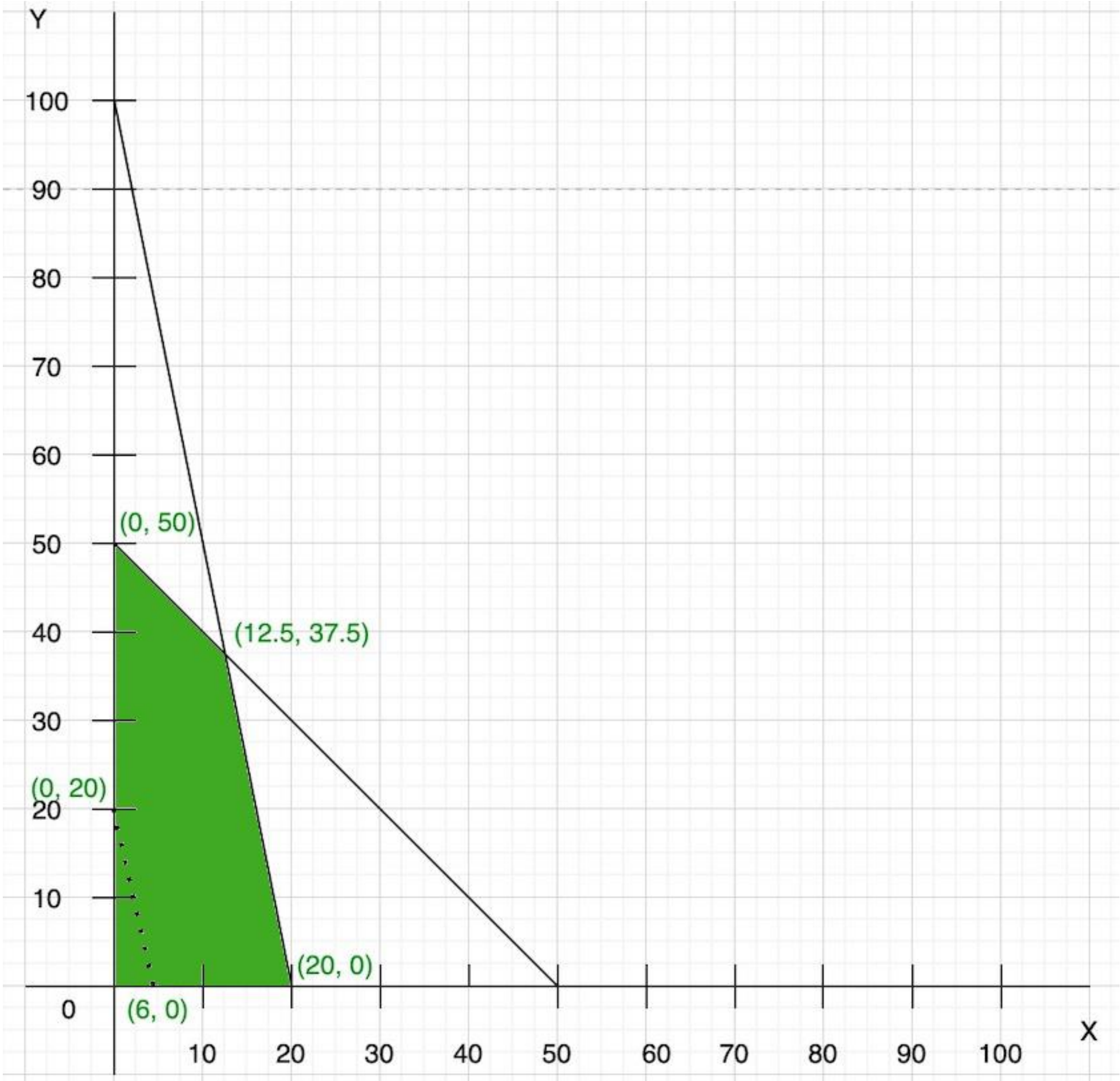
$50x + 15y = 300$

Now we find the points

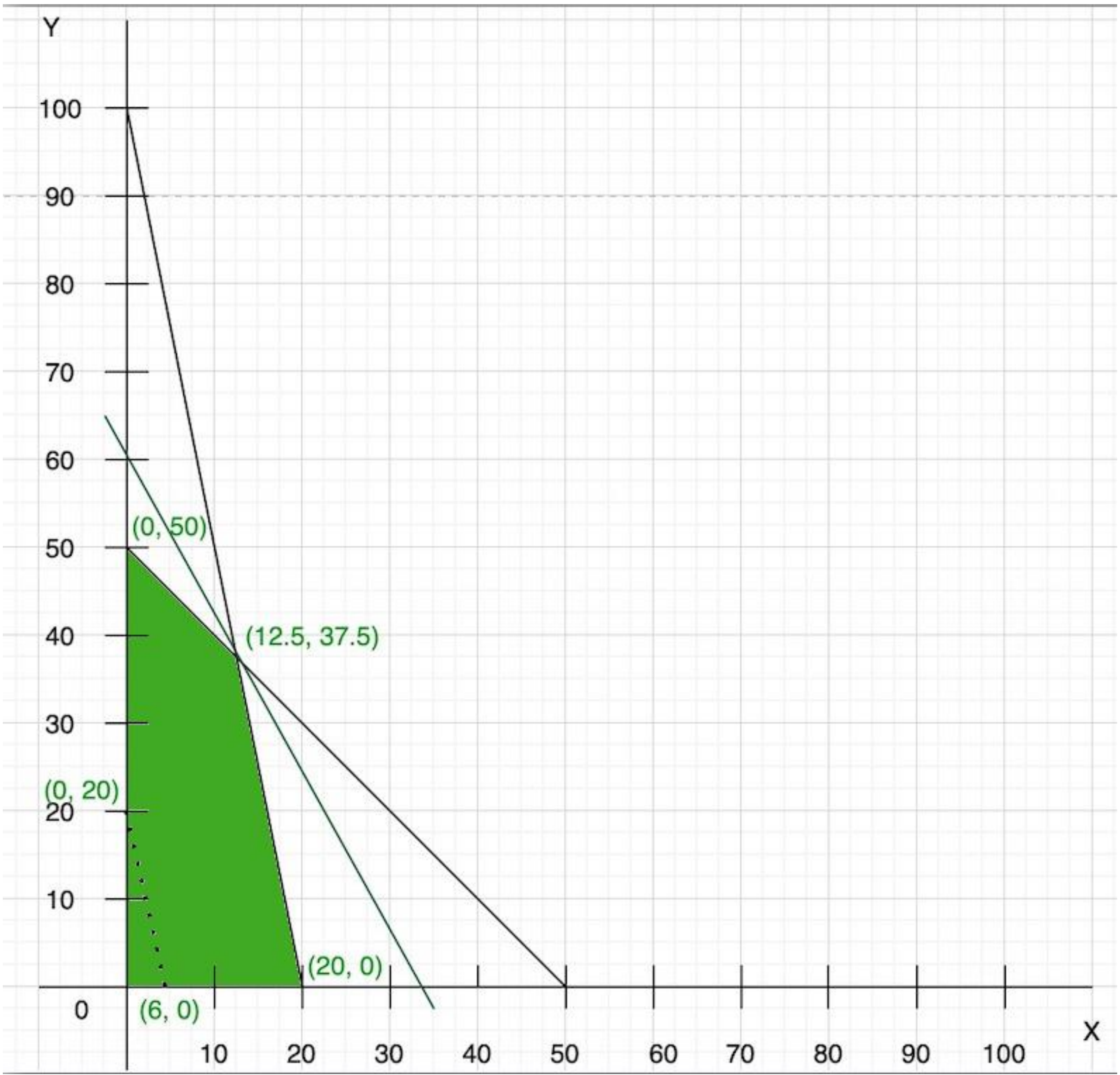
Put $x = 0$, $y = 20$

Put $y = 0$, $x = 6$

draw the line of this objective function on the graph:



Step 4: As we know that the objective function is maximum type then we draw a line which is parallel to the objective function line and farthest from the origin and only has one common point to the feasible region.



Step 5: We have a common point that is (12.5, 37.5) with the feasible region. So, now we find the optimal solution of the objective function:

Z = 50x + 15y

Z = 50(12.5) + 15(37.5)

Z = 625 + 562.5

Z = 1187

Thus, maximum value of Z with given constraint is, 1187

<u>Simplex and Dual Simplex Method</u>

The **Simplex method** is a popular algorithm used to solve linear programming problems. It starts with a basic feasible solution and works towards optimality. If the primal problem is too difficult to solve, then it is easier to solve the dual problem. The **Dual Simplex method** starts with an infeasible solution which is optimal and works towards optimality. It can be roughly described as the simplex method applied to the dual linear program. The Dual Simplex method performs well in case of non-degeneracy.

Computational Procedure of Dual Simplex Method

Introduction:

Any LPP for which it is possible to find infeasible but better than optimal initial basic solution
can be solved by using dual simplex method. Such a situation can be recognized by first
expressing the constraints in '≤' form and the objective function in the maximization form. After
adding slack variables, if any right hand side element is negative and the optimality condition is
satisfied then the problem can be solved by dual simplex method.
Negative element on the right hand side suggests that the corresponding slack variable is
negative. This means that the problem starts with optimal but infeasible basic solution and we
proceed towards its feasibility.
The dual simplex method is similar to the standard simplex method except that in the latter the
starting initial basic solution is feasible but not optimum while in the former it is infeasible but
optimum or better than optimum. The dual simplex method works towards feasibility while
simplex method works towards optimality

Computational Procedure of Dual Simplex Method

The iterative procedure is as follows

Step 1 - First convert the minimization LPP into maximization form, if it is given in the minimization form.

Step 2 - Convert the '≥' type inequalities of given LPP, if any, into those of '≤' type by multiplying the corresponding constraints by -1.

Step 3 – Introduce slack variables in the constraints of the given problem and obtain an initial
basic solution.

Step 4 – Test the nature of Δj in the starting table

- If all Δj and XB are non-negative, then an optimum basic feasible solution has been attained.

- If all Δj are non-negative and at least one basic variable XB is negative, then go to step 5.
- If at least Δj one is negative, the method is not appropriate.

Step 5 – Select the most negative XB. The corresponding basis vector then leaves the basis set B.
Let Xr be the most negative basic variable.
Step 6 – Test the nature of Xr

- If all Xr are non-negative, then there does not exist any feasible solution to the given problem.
- If at least one Xr is negative, then compute Max (Δj / Xr) and determine the least negative for incoming vector.

Step 7 – Test the new iterated dual simplex table for optimality.
Repeat the entire procedure until either an optimum feasible solution has been attained in a finite number of steps.

13.3 Worked Examples
Example 1
Minimize Z = 2x1 + x2
Subject to
3x1 + x2 ≥ 3
4x1 + 3x2 ≥ 6
x1 + 2x2 ≥ 3
and x1 ≥ 0, x2 ≥ 0
Solution
Step 1 – Rewrite the given problem in the form
Maximize Z 2 – = ′ x1 – x2
Subject to
–3x1 – x2 ≤ –3
–4x1 – 3x2 ≤ –6
–x1 – 2x2 ≤ –3
x1, x2 ≥ 0
Step 2 – Adding slack variables to each constraint
Maximize Z 2 – = ′ x1 – x2
Subject to
–3x1 – x2 + s1 = –3
–4x1 – 3x2 + s2 = –6
–x1 – 2x2 + s3 = –3
x1, x2, s1,s2, s3 ≥ 0
Step 3 – Construct the simplex table

	$C_j \rightarrow$		-2	-1	0	0	0	
Basic variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	
s_1	0	-3	-3	-1	1	0	0	
s_2	0	-6	-4	[-3]	0	1	0	→ outgoing
s_3	0	-3	-1	-2	0	0	1	
	$Z' = 0$		2	↑ 1	0	0	0	$\leftarrow \Delta_j$

Step 4 – To find the leaving vector
Min (-3, -6, -3) = -6. Hence s2 is outgoing vector
Step 5 – To find the incoming vector
Max (Δ1 / x21, Δ2 / x22) = (2/-4, 1/-3) = -1/3. So x2 is incoming vector
Step 6 –The key element is -3. Proceed to next iteration

	$C_j \rightarrow$		-2	-1	0	0	0	
Basic variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	
s_1	0	-1	-5/3	0	1	-1/3	0	→ outgoing
x_2	-1	2	4/3	1	0	-1/3	0	
s_3	0	1	5/3	0	0	-2/3	1	
	$Z' = -2$		↑ 2/3	0	0	1/3	0	$\leftarrow \Delta_j$

Step 7 – To find the leaving vector
Min (-1, 2, 1) = -1. Hence s1 is outgoing vector
Step 8 – To find the incoming vector
Max (Δ1 / x11, Δ4 / x14) = (-2/5, -1) = -2/5. So x1 is incoming vector
Step 9 –The key element is -5/3. Proceed to next iteration

	$C_j \rightarrow$		-2	-1	0	0	0	
Basic variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	
x_1	-2	3/5	1	0	-3/5	1/5	0	
x_2	-1	6/5	0	1	4/5	-3/5	0	
s_3	0	0	0	0	1	-1	1	
	$Z' = -12/5$		0	0	2/5	1/5	0	$\leftarrow \Delta_j$

Step 10 – Δj ≥ 0 and XB ≥ 0, therefore the optimal solution is Max Z -12/5 = ′ , Z = 12/5, and x1=3/5, x2 = 6/5

Example 2
Minimize Z = 3x1 + x2
Subject to
x1 + x2 ≥ 1
2x1 + 3x2 ≥ 2
and x1 ≥ 0, x2 ≥ 0
3
Solution
Maximize Z 3 – = ′ x1 – x2
Subject to
–x1 – x2 ≤ –1
–2x1 – 3x2 ≤ –2
x1, x2 ≥ 0
SLPP
Maximize Z 3 – = ′ x1 – x2
Subject to
–x1 – x2 + s1 = –1
–2x1 – 3x2 + s2 = –2
x1, x2, s1,s2 ≥ 0

	$C_j \rightarrow$		-3	-1	0	0	
Basic variables	C_B	X_B	X_1	X_2	S_1	S_2	
s_1	0	-1	-1	-1	1	0	
s_2	0	-2	-2	-3	0	1	$\rightarrow$
	$Z' = 0$		3	$\uparrow$ 1	0	0	$\leftarrow \Delta_j$
s_1	0	-1/3	-1/3	0	1	-1/3	$\rightarrow$
x_2	-1	2/3	2/3	1	0	-1/3	
	$Z' = -2/3$		7/3	0	0	$\uparrow$ 1/3	$\leftarrow \Delta_j$
s_2	0	1	1	0	-3	1	
x_2	-1	1	1	1	-1	0	
	$Z' = -1$		2	0	1	0	$\leftarrow \Delta_j$

Δj ≥ 0 and XB ≥ 0, therefore the optimal solution is Max Z -1 = ′ , Z = 1, and x1= 0, x2= 1

Sensitivity Analysis

Sensitivity analysis is a method used to study how changes in one or more input variables affect the output variables of a mathematical model or system. It can help predict the outcome of a decision or a behaviour under different scenarios or assumptions. It can also be used to test the effectiveness of antibiotics against bacteria. Sensitivity analysis is often used in financial modeling to determine how target variables are affected based on changes in other variables known as input variables.

In sensitivity analysis, one looks at the effect of varying the inputs of a mathematical model on the output of the model itself. The process of recalculating outcomes under alternative assumptions to determine the impact of a variable under sensitivity analysis can be useful for a range of purposes, including testing the robustness of the results of a model or system in the presence of uncertainty, increased understanding of the relationships between input and output variables in a system or model, uncertainty reduction, through the identification of model input that cause significant uncertainty in the output and should therefore be the focus of attention in order to increase robustness (perhaps by further research), searching for errors in the model (by encountering unexpected relationships between inputs and outputs), model simplification – fixing model input that has no effect on the output, or identifying and removing redundant parts of the model structure, enhancing communication from modelers to decision makers (e.g. by making recommendations more credible, understandable, compelling or persuasive), finding regions in the space of input factors for which the model output is either maximum or minimum or meets some optimum criterion (see optimization and Monte Carlo filtering).

Transportation and Assignment Models

Transportation and assignment models are special purpose algorithms of linear programming. They are used to solve problems like determining the optimum assignment of jobs to persons, supply of materials from several supply points to several destinations, and the like. The transportation model is concerned with selecting the routes between supply and demand points in order to minimize costs of transportation subject to constraints of supply at any supply point and demand at any demand point. The assignment model is used when there are facilities and jobs which have to be assigned to those facilities. Unlike a transportation model, in an assignment model, the number of facilities (sources) is equal to the number of jobs (destinations). The transportation algorithm can be used to solve the assignment model, but the assignment algorithm cannot be used to solve the transportation model.

Hungarian algorithm

The **Hungarian method** is a combinatorial optimization algorithm that solves the assignment problem in polynomial time and which anticipated later primal–dual methods.

Core of the algorithm (assuming square matrix):

1. For each row of the matrix, find the smallest element and subtract it from every element in its row.
2. Do the same (as step 1) for all columns.
3. Cover all zeros in the matrix using minimum number of horizontal and vertical lines.
4. *Test for Optimality:* If the minimum number of covering lines is n, an optimal assignment is possible and we are finished. Else if lines are lesser than n, we haven't found the optimal assignment, and must proceed to step 5.
5. Determine the smallest entry not covered by any line. Subtract this entry from each uncovered row, and then add it to each covered column. Return to step 3.

Example:

Four people A,B,C,D Four Job J1,J2,J3,J4

	J1	J2	J3	J4
A	10	7	8	2
B	1	5	6	3
C	2	10	3	9
D	4	3	2	3

The goal is to assign people to Job in way that minimum cost.

Answer:

Below is the cost matrix of example given in above

10	7	8	2
1	5	6	3
2	10	3	9
4	3	2	3

Step 1: Subtract minimum of every row.

8	5	6	0
0	4	5	2
0	8	1	7
2	1	0	1

Step 2: Subtract minimum of every column.

8	4	6	0
0	3	5	2
0	7	1	7
2	0	0	1

Step 3: Cover all zeroes with minimum number of horizontal and vertical lines.

8	4	6	0
0	3	5	2
0	7	1	7
2	0	0	1

Step 4: . If the number of these straight lines zero is less than n, meaning that the solution is not optimal and needs to be developed.
Choosing the smallest element that was not passed by any line of straight lines and adding it to every element intersects the two lines and subtracting it on all the elements that were not passed by any line.

8	4	6	0
0	3	5	2
0	7	1	7
2	0	0	1

⇓

9	4	6	0
0	2	4	1
0	6	0	6
3	0	0	1

Step 5:

	J1	J2	J3	J4
A	9	4	6	0
B	0	2	4	1
C	0	6	0	6
D	3	0	0	1

Total Cost :

A→J4, B→J1, C→J3, D→J2

	J1	J2	J3	J4
A	10	7	8	2
B	1	5	6	3
C	2	10	3	9
D	4	3	2	3

Original cost matrix

= 2+1+3+3

= 9

PERT vs CPM

Abbreviation	
PERT – Project Evaluation and Review Technique	CPM – Critical Path Method
What does It Mean?	
PERT – PERT is a popular project management technique that is applicable when the time required to finish a project is not certain	CPM – CPM is a statistical algorithm which has a certain start and end time for a project
Model Type	
PERT – PERT is a probabilistic model	CPM – CPM is a deterministic model
Focus	
PERT – The main focus of PERT is to minimise the time required for completion of the project	CPM – The main focus of CPM is on a trade-off between cost and time, with a major emphasis on cost-cutting.
Orientation type	
PERT – PERT is an event-oriented technique	CPM – CPM is an activity-oriented technique

What is the critical path method (CPM)?

The critical path method (CPM) is a project management technique that's used by project managers to create an accurate project schedule. The CPM method, also known as critical path analysis (CPA), consists in using the CPM formula and a network diagram to visually represent the task sequences of a project. Once these task sequences or paths are defined, their duration is calculated to identify the critical path.

Critical Path Method (CPM) Formula

Before we learn how to use the CPM formula, we need to understand some key CPM concepts.

- Earliest start time (ES): This is simply the earliest time that a task can be started in your project. You cannot determine this without first knowing if there are any task dependencies
- Latest start time (LS): This is the very last minute in which you can start a task before it threatens to delay your project timeline
- Earliest finish time (EF): The earliest an activity can be completed, based on its duration and its earliest start time
- Latest finish time (LF): The latest an activity can be completed, based on its duration and its latest start time
- Float: Also known as slack, float is a term that describes how long you can delay a task before it impacts its task sequence and the project schedule. The tasks on the critical path have zero float because they can't be delayed

- The critical path method formula has two parts; a forward pass and a backward pass.
- Forward Pass in CPM
- Use the CPM diagram and the estimated duration of each activity to determine their earliest start (ES) and earliest finish (EF). The ES of an activity is equal to the EF of its predecessor, and its EF is determined by the CPM formula EF = ES + t (t is the activity duration). The EF of the last activity identifies the expected time required to complete the entire project.

- Backward Pass in CPM

- Begins by assigning the last activity’s earliest finish as its latest finish. Then the CPM formula to find the LS is LS = LF – t (t is the activity duration). For the previous activities, the LF is the smallest of the start times for the activity that immediately follows.

What Is Resource Leveling?

It is a technique in project management that resolves various conflicts, such as schedule conflicts or over or under-allocation of resources, to ensure that the available resources can be utilized to their fullest extent and the project gets completed at the earliest.

Resource leveling is mainly done by setting realistic project deadlines by extending or curbing a project's start and finish dates. It helps maintain the project's cost while not forcing the employees to overwork.

Why Is It Important?

To Optimize Your Resources

To Minimize Deficits

To Prevent Task Overloading

To Ensure the Quality of a Project Output

How Do Resource Allocation, Resource Leveling, and Resource Smoothing Work Together?

Project management can be efficiently optimized by combining resource allocation, smoothing and leveling.

When you allocate tasks to individual team members within their area of strength to ensure the work gets done by them, that is called resource allocation.

By allotting daily buffers, resource smoothing helps distribute the workload evenly among the available resources.

Leveling resources promotes a balanced work-life ratio by ensuring no employees work more than the regular hours and do not exceed their limits.

Resource allocation helps by mapping out the project and tackling budgeting, resource planning, and project outcome during the initial planning phase. It helps tackle any unforeseen problems that might arise during the project. Resource smoothening balances the high and low workload to ensure the resources do not exceed their limits.

Cost Consideration in Project Scheduling

Cost considerations in project scheduling refer to the **money and resources** required to complete a project. This includes equipment, people, and materials. The customer usually wants the project completed at the lowest cost possible, and the budget for the project is approved based on the scope and schedule.

Effective **cost management** is essential for project management success. Ineffective cost management is often the very reason that projects fail. The cost management process begins in the planning phase of the project, where costs are estimated and then a project budget is defined. Then, when the project is executed, the expenses are carefully monitored and recorded to make sure that they're aligned with the budget.

Project scheduling is just as important as cost budgeting as it determines the timeline, resources needed, and reality of the delivery of the project. Project managers that have experience are better able to properly dictate the tasks, effort and money required to complete a project.

Vogel's Approximation Method (VAM)

Vogel's Approximation Method (VAM) is one of the methods used to calculate the initial basic feasible solution to a transportation problem. However, VAM is an iterative procedure such that in each step, we should find the penalties for each available row and column by taking the least cost and second least cost.

Vogel's Approximation Method Steps

Below are the steps involved in Voge's approximation method of finding the feasible solution to a transportation problem.

Step 1: Identify the two lowest costs in each row and column of the given cost matrix and then write the absolute row and column difference. These differences are called penalties.

Step 2: Identify the row or column with the maximum penalty and assign the corresponding cell's min(supply, demand). If two or more columns or rows have the same maximum penalty, then we can choose one among them as per our convenience.

Step 3: If the assignment in the previous satisfies the supply at the origin, delete the corresponding row. If it satisfies the demand at that destination, delete the corresponding column.

Step 4: Stop the procedure if supply at each origin is 0, i.e., every supply is exhausted, and demand at each destination is 0, i.e., every demand is satisfying. If not, repeat the above steps, i.e., from step 1.

Vogel's Approximation Method Solved Example

Factories	Destination centers				Supply
	D_1	D_2	D_3	D_4	
F_1	3	2	7	6	50
F_2	7	5	2	3	60
F_3	2	5	4	5	25
Demand	60	40	20	15	

Solution:

For the given cost matrix,

Total supply = 50 + 60 + 25 = 135

Total demand = 60 + 40 + 20 + 25 = 135

Thus, the given problem is balanced transportation problem.

Now, we can apply the Vogel's approximation method to minimize the total cost of transportation.

Step 1: Identify the least and second least cost in each row and column and then write the corresponding absolute differences of these values. For example, in the first row, 2 and 3 are the least and second least values, their absolute difference is 1.

	D_1	D_2	D_3	D_4	Supply	Row difference
F_1	3	2	7	6	50	1
F_2	7	5	2	3	60	1
F_3	2	5	4	5	25	2
Demand	60	40	20	15		
Column difference	1	3	2	2		

These row and column differences are called penalties.

Step 2: Now, identify the maximum penalty and choose the least value in that corresponding row or column. Then, assign the min(supply, demand).

Here, the maximum penalty is 3 and the least value in the corresponding column is 2. For this cell, min(supply, demand) = min(50, 40) = 40

Allocate 40 in that cell and strike the corresponding column since in this case demand will be satisfied, i.e., 40 – 40 = 0.

	D_1	D_2	D_3	D_4	Supply	Row difference
F_1	3	40 2	7	6	50 - 40=10	1
F_2	7	5	2	3	60	1
F_3	2	5	4	5	25	2
Demand	60	40-40=0	20	15		
Column difference	1	3	2	2		

Step 3: Now, find the absolute row and column differences for the remaining rows and columns. Then repeat step 2.

Here, the maximum penalty is 3 and the least cost in that corresponding row is 3. Also, the min(supply, demand) = min(10, 60) = 10

Thus, allocate 10 for that cell and write down the new supply and demand for the corresponding row and column.

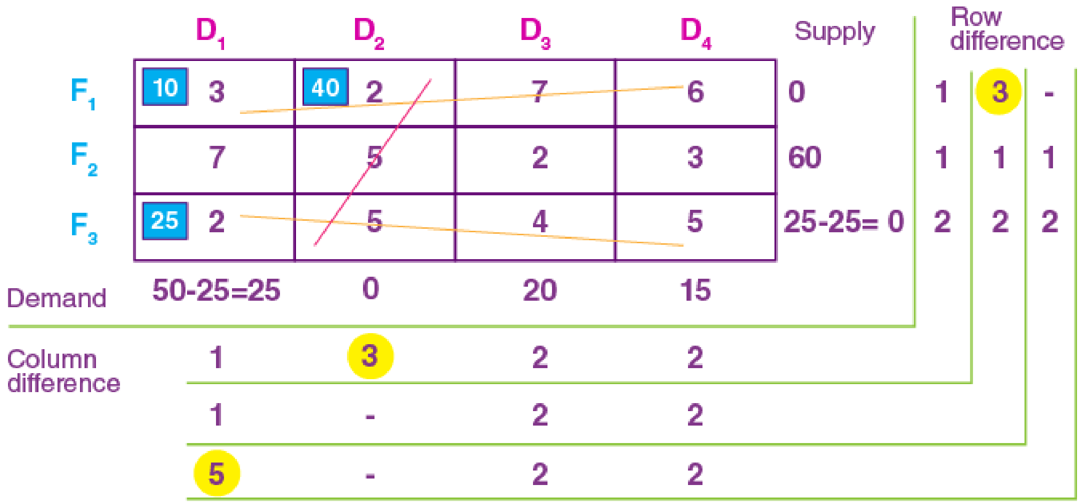
Supply = 10 – 10 = 0

Demand = 60 – 10 = 50

As supply is 0, strike the corresponding row.

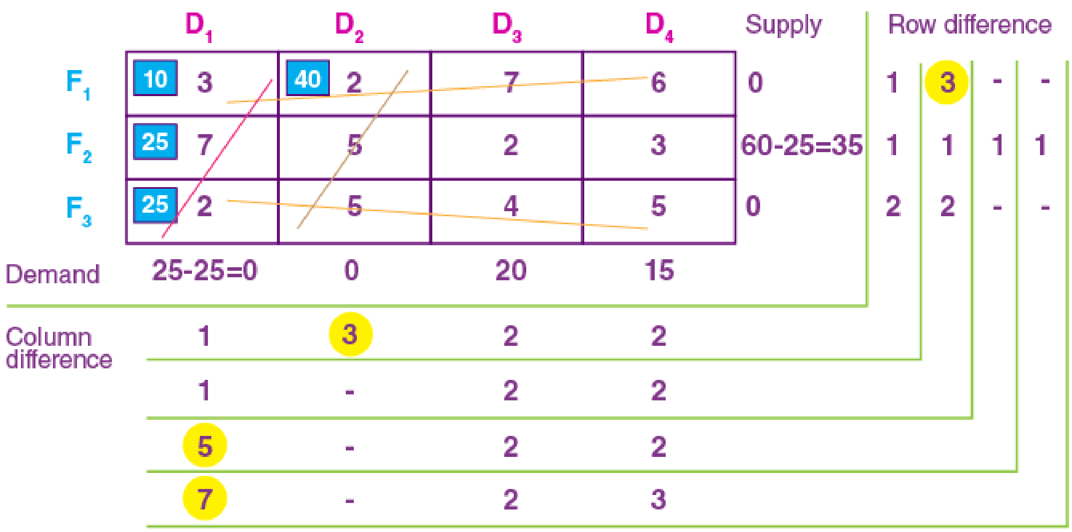
	D_1	D_2	D_3	D_4	Supply	Row difference	
F_1	10 3	40 2	7	6	10-10=0	1	3
F_2	7	5	2	3	60	1	1
F_3	2	5	4	5	25	2	2
Demand	60-10=50	0	20	15			
Column difference	1	3	2	2			
	1	-	2	2			

Step 4: Repeat the above step, i.e., step 3. This will give the below result.



In this step, the second column vanishes and the min(supply, demand) = min(25, 50) = 25 is assigned for the cell with value 2.

Step 5: Again repeat step 3, as we did for the previous step.



In this case, we got 7 as the maximum penalty and 7 as the least cost of the corresponding column.

Step 6: Now, again repeat step 3 by calculating the absolute differences for the remaining rows and columns.

	D1	D2	D3	D4	Supply	Row difference				
F1	[10] 3	[40] 2	7	6	0	1	3	-	-	-
F2	[25] 7	5	2	[15] 3	35-15=20	1	1	1	1	1
F3	[25] 2	5	4	5	0	2	2	-	-	-
Demand	25-25=0	0	20	15-15=0						
Column difference	1	3	2	2						
	1	-	2	2						
	5	-	2	2						
	7	-	2	3						
	-	-	2	3						

Step 7: In the previous step, except for one cell, every row and column vanishes. Now, allocate the remaining supply or demand value for that corresponding cell.

	D1	D2	D3	D4	Supply	Row difference				
F1	[10] 3	[40] 2	7	6	0	1	3	-	-	-
F2	[25] 7	5	[20] 2	[15] 3	20-20=0	1	1	1	1	1
F3	[25] 2	5	4	5	0	2	2	-	-	-
Demand	25-25=0	0	20-20=0	0						
Column difference	1	3	2	2						
	1	-	2	2						
	5	-	2	2						
	7	-	2	3						
	-	-	2	3						

Total cost = (10 × 3) + (25 × 7) + (25 × 2) + (40 × 2) + (20 × 2) + (15 × 3)

= 30 + 175 + 50 + 80 + 40 + 45

= 420